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ON THE TOPOLOGICAL CLASSIFICATION OF LINEAR REPRESENTATIONS

REINHARD SCHULTZ

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GIVEN a compact Lie group, a basic mathematical problem is the classification of its finite-dimensional orthogonal representations up to linear equivalence. Recently such representations have been classified up to certain homotopy-theoretic relations much weaker than linear equivalence[13, 16]. In this paper we shall study the classification up to an intermediate relation—topological equivalence; to be precise, two orthogonal representations $\{V, \rho_V\}$ and $\{W, \rho_W\}$ (ρ_X: $G \to 0(X)$ denotes the representation homomorphism) are topologically equivalent if there is a homeomorphism h: $V \to W$ such that $\rho_W = h\rho_V h^{-1}$ (i.e., V and W are equivariantly homeomorphic). The main result states that topological equivalence agrees with linear equivalence for a large class of compact Lie groups (see Theorem I below together with Corollary 2.3 and (2.4)); in fact, no examples are known for which the classifications differ.

The results of this paper were obtained to answer questions posed by R. Lashof and M. Rothenberg; the topological equivalence question for representations arises naturally in their equivariant smoothing theory for topological G-manifolds. Shortly after the first version of this paper was written, the author discovered from [12, §9δ] that D. Sullivan had worked out the principal part of the main result some time ago but had not written up his conclusions for publication.

The main result may be stated as follows:

THEOREM I. *Let G be a compact Lie group with* p^k *components, where p is an odd prime and* $0 \leq k \in \mathbf{Z}$. *Then two G-representations are topologically equivalent if and only if they are linearly equivalent.*

The proof of Theorem I quickly reduces to proving that two vector bundles over the classifying space BG are inequivalent as topological Euclidean space bundles (see §2 for specifics). Sullivan's characterization of such bundles by KO-orientations away from the prime 2 [21, §6] or [15, Ch. V] provides a powerful method for studying this problem, which is developed in §1. We shall state the principal conclusion in the following form suggested by M. F. Atiyah.

THEOREM II. *Let G be a finite group, and let KTop denote Milnor's K-theory of topological Euclidean space bundles*[17]. *Then the forgetful map*

$$\alpha_G: KO(BG) \to KTop(BG)$$

is a monomorphism if G has no elements of order 8. *Conversely, if G has a cyclic central subgroup H of order* 8, *then* α_G *is not a monomorphism.*

If G is cyclic or even p-elementary for p odd, this result gives necessary and sufficient conditions for α_G to be monic.

At the end of §2 we shall show that the conclusion of Theorem I is valid for certain groups not covered in the hypothesis. The smallest group untreated by our methods is $\mathbf{Z}/8\mathbf{Z}$.

As noted previously, Theorem I has applications to the classification of smooth G-manifolds that are topologically equivalent. In the last section of this paper we shall prove one simple result of this sort: For suitable finite groups G, the differentiable slice representations at points are topologically invariant (see Theorem III). This greatly simplifies the problem of classifying certain smooth actions topologically.

§1. THE MAP $KO(BG) \to KTop(BG)$

Given a vector bundle ξ over X, Sullivan defines a $KO \otimes \mathbf{Z}[\frac{1}{2}]$-orientation Δ_ξ of ξ in [21, §6] and proves its topological invariance. Given this orientation, topologically invariant characteristic classes $\theta^k(\xi)\in KO^*(X)\otimes \mathbf{Z}[\frac{1}{2}]$ are defined for each positive integer† k by the formula $\theta^k(\xi)\Delta_\xi = \psi^k(\Delta_\xi)$ (e.g., see [15, Ch. V] or [21, §6]), and they have the following properties.

$$\theta^k(\xi) \text{ is a multiplicative unit.} \tag{1.1}$$

$$\theta^k(\xi_1 \oplus \xi_2) = \theta^k(\xi_1)\cdot\theta^k(\xi_2). \tag{1.2}$$

If η is a complex line bundle, then in $KO^*(X)\otimes\mathbf{Z}[\frac{1}{2}, k^{-1}]$ one has

$$\theta^k(\mathbf{re}\,\eta)\otimes \mathbf{C} = \frac{1(\eta^{2k}-1)^2(\eta^4-1)}{k(\eta^2-1)^2(\eta^{4k}-1)} \tag{1.3}$$

(see [21, pp. 6.71, 6.82]; the right hand side is in the image of complexification because it is invariant under conjugation).

Suppose p is an odd prime and $r(p)$ is a primitive root of 1 mod p^2. Then ξ maps to zero in

$$KTop^{\sim}(X)\otimes \mathbf{Z}_{(p)} \tag{1.4}$$

if and only if $\theta^{r(p)}(\xi)$ maps to 1 in $KO^*(X)\otimes\mathbf{Z}_{(p)}$ (compare [15, Ch.V, Prop. 6.13] or [24, Thm. 6.9, p. 6.73]).

The calculation of $K(BG)$ and $KO(BG)$ for G finite is essentially due to Atiyah[3, 4] and Anderson[2]: If $R(G)(RO(G))$ denotes the complex (resp., real) representation right of G, $I(G)$ $(IO(G))$ is the ideal of elements with virtual dimension zero, and $R(G)^\wedge$ $(RO(G)^\wedge)$ is the $I(G)$-adic $(IO(G)$-adic$)$ completion, then $K(BG)$ $(KO(BG))$ is isomorphic to the completed ring $R(G)^\wedge$ $(RO(G)^\wedge)$. The map $R(G)\to K(BG)$ $(RO(G)\to KO(BG))$ is given by the balanced product construction. If G has odd order, then $IO(G)^\wedge$ is canonically isomorphic to the self-conjugate part of $I(G)^\wedge$.

Of course, $R(\,)$ is additively a free abelian group; it will be convenient to have a similar description of $R(G)^\wedge$, at least if G is cyclic of prime power order. Since $R(G)^\wedge \cong \mathbf{Z}\oplus I(G)^\wedge$ additively, it suffices to describe $I(G)^\wedge$.

PROPOSITION 1.5. *Let p be a prime, let G be cyclic of order $p^s (s\geq 1)$, and let t denote the standard algebra generator of $R(G)\cong \mathbf{Z}[t]/(t^{|G|}-1)$. Then $I(G)^\wedge$ is topologically isomorphic (additively) to the free module over the p-adic integers generated by $\{t^i-1 \mid 1\leq i<|G|\}$.*

This result follows because the $I(G)$-adic and p-adic filtrations of $I(G)$ give the same completion. In particular, the relation $(1+\sigma)^{p^s}=1$ (where $\sigma = t-1$) implies $p^sI\subseteq I^2$ and $I^{p^s}\subseteq pI$.

It will be convenient to split $I(G)^\wedge$ into $I_\gamma(G)^\wedge\oplus I_\delta(G)^\wedge$, where $I_\gamma(G)^\wedge$ is the (closed) subgroup freely generated by the elements t^i-1 with $(p,i)=1$ and $I_\delta(G)^\wedge$ is the (closed) subgroup freely generated by all t^i-1 with $i\equiv 0 \bmod p$. Observe that $I_\delta(G)^\wedge$ is isomorphic to $I(G/p^{s-1}G)^\wedge$; in fact, if π: $G\to G/p^{s-1}G$ is the obvious projection, this isomorphism is induced by π^*.

The above notation allows us to isolate the most difficult part of the proof of Theorem II in the following form:

PROPOSITION 1.6. *Let G be cyclic of order p^s, where p is an odd prime. Then the restriction of α_G: $KO(BG)\to KTop(BG)$ to $I_\gamma(G)^\wedge\cap KO(BG)\subseteq K(BG)$ is a monomorphism.*

Notation. Objects of the form $I\cap KO$ will be denoted by IO.

Proof. Let $\mathbf{Q}_p$ be the p-adic numbers, $\mathbf{Q}_p[\xi]$ its cyclotomic extension by a $|G|$-th root of 1, and define a homomorphism

$$\gamma\colon\ I(G)^\wedge\to\mathbf{Q}_p[\omega]$$

taking $\Sigma\, a_i(t^i-1)$ $(a_i\in\mathbf{Z}^\wedge_{(p)}$, the p adic integers) to $\Sigma\, a_i \log_{(p)}\theta^r(\xi^i-1)$, where θ^r is defined as in (1.3) and (1.4), and $\log_{(p)}$ denotes the p-adic logarithm.‡ According to (1.4), Kernel $\lambda\supseteq$ Kernel $\alpha_G\cap IO_\gamma(G)^\wedge$. Therefore the necessary conditions for $\Sigma\, a_i(t^i-1)$ to lie in the latter set are (i) $a_i = a_{-i}$ (by self-conjugacy) and (ii) $\Sigma\, a_i\log_{(p)}\theta^r(\xi^i-1)=0$. We claim this can happen only if each

†In fact, they are definable for each "$\frac{1}{2}$-adic" integer in the profinite completion of $KO^*(X)\otimes\mathbf{Z}[\frac{1}{2}]$, but we shall not need this.
‡See [7, p. 121] for the definition.

$a_j = 0$. Taking exponentials, we may rewrite (ii) in the form

$$\prod \left(\frac{1}{r}\right)^{a_j}\left(\frac{\xi^{2rj}-1}{\xi^{2j}-1}\right)^{2a_j}\left(\frac{\xi^{4j}-1}{\xi^{4rj}-1}\right)^{a_j} = 1, \tag{1.7}$$

the product running over all $j \in (\mathbf{Z}/|G|\mathbf{Z})^*$, the unit group of $\mathbf{Z}/|G|\mathbf{Z}$.

First of all, we claim that $\Sigma\, a_j = 0$; this will allow us to delete the factors r^{-a_j}. To prove the assertion take the product of all the conjugates of (1.7) under the action of the Galois group of $\mathbf{Q}_p[\xi]/\mathbf{Q}_p$. The resulting equation reduces to $1 = r^{-\varphi\Sigma a_j}$, where φ is the order of the Galois group; but this implies $\Sigma\, a_j = 0$.

Grouping together all factors of the form $\xi^k - 1$, we may rewrite (1.7) as

$$\prod_{k\in(\mathbf{Z}/|G|\mathbf{Z})} (\xi^k - 1)^{b_k} = 1, \quad \text{where} \qquad b_k = 2a_{k/2r} + a_{k/4} - 2a_{k/2} - a_{k/4r}. \tag{1.8}$$

We claim that each b_k is equal to zero. If the b_k are all assumed to be ordinary integers, this result is due to W. Franz (see [10, p. 1–12] for a proof) and has been used in the past to classify lens spaces and free G-representations up to topological equivalence ([10], [9, §31 and Appendix], and [12, §9]). On the other hand, Franz' theorem and a result of A. Brumer [7, Thm. 2] combine to show that the p-adic logarithms

$$\log_{(p)}\left(\frac{\xi^j-1}{\xi-1}\right) \quad 1 < j < \frac{|G|}{2}$$

are linearly independent over the p-adic numbers, and thus each $b_k = 0$ even if the b_k are merely assumed to be p-adic integers.

The equations $b_k = 0$ for all k may be rewritten in the form

$$2a_{k/2r} + a_{k/4} = 2a_{k/2} + a_{k/4r}.$$

Thus if we set $c_l = a_l - a_{l/r}$, the equation may be again rewritten as $c_l = 2c_{2l}$. Choose $e \geq 1$ so that $2^e \equiv 1 \bmod p^s$. Then $c_l = 2^e c_{2^e l} = 2^e c_l$ by induction; since $2^e \neq 1$, this means $c_l = 0$ for all l. It follows that $a_{rk} = a_k$ for all k; since r generates the unit group of $\mathbf{Z}/p^s\mathbf{Z}$, all the a_k's must be equal; on the other hand, the a_k's add up to zero, and these conditions are consistent only if each $a_k = 0$. This completes the proof of Proposition 1.6.

Proof of Theorem II

Sufficiency. We shall first prove that α_G is monic if $G = \mathbf{Z}/p^s\mathbf{Z}$ (p odd) by induction on s. The case $s = 0$ is trivial, so assume α is monic for $\mathbf{Z}/p^{s-1}\mathbf{Z}$. Let $i = \mathbf{Z}/p^{s-1}\mathbf{Z} \to \mathbf{Z}/p^s\mathbf{Z}$ and π: $\mathbf{Z}/p^s\mathbf{Z} \to \mathbf{Z}/p^{s-1}\mathbf{Z}$ be the canonical injection and projection respectively.

Given an integer k satisfying $1 \leq k \leq p-1$, let φ_k be the automorphism of $\mathbf{Z}/p^s\mathbf{Z}$ taking 1 to $kp^{s-1}+1$. Consider the map φ_k^* induced by φ_k on $KO^{\sim}(BG) \cong IO_\gamma(G)^{\wedge} \oplus IO_\delta(G)^{\wedge}$; clearly φ_k^* maps both summands to themselves, fixes the second summand, and maps $t-1$ to $t^{1+kp^{s-1}} - 1$.

Let $x \in KO^{\sim}(BG)$, and express $x = x_\gamma + x_\delta$, where $x_\gamma \in IO_\gamma^{\wedge}$ and $x_\delta \in IO_\delta^{\wedge}$. Assume $x \in$ Kernel α_G. By naturality $\varphi_k^* x$ also lies in Kernel α_G, and hence $\varphi_k^* x_\gamma - x_\gamma = \varphi_k^* x - x \in$ Kernel α_G. By proposition 1.6, this means $\varphi_k^* x_\gamma = x_\gamma$ must hold for all k. It follows that x_γ may be written as a double sum

$$x_\gamma = \sum_{\substack{j=1\\(j,p)=1}}^{p^{s-1}} \sum_{k=0}^{p-1} a_j(t^{j+kp^{s-1}}). \tag{1.9}$$

Consider the element $i^*x = i^*x_\gamma + i^*x_\delta$; it is immediate that $i^*x \in$ Kernel α, $i^*x_\gamma \in IO_\gamma(\mathbf{Z}/p^{s-1}\mathbf{Z})^{\wedge}$ and $i^*x_\delta \in IO_\delta(\mathbf{Z}/p^{s-1}\mathbf{Z})^{\wedge}$. Furthermore by (1.9) we may write $i^*x_\gamma = py$, where $y = \Sigma\, a_j(t^j - 1) \in IO_\gamma(\mathbf{Z}/p^{s-1}\mathbf{Z})^{\wedge}$. By the induction hypothesis, $0 = i^*x = i^*x_\delta + i^*x_\gamma = py$. Since $IO(\mathbf{Z}/p^{s-1}\mathbf{Z})^{\wedge}$ is a free $\mathbf{Z}^{\wedge}_{(p)}$ module and the latter has no torsion, this implies each $a_j = 0$, which in turn implies $x_\gamma = 0$. On the other hand, $x_\delta = \pi^*z$ for some $z \in KO \sim (B\mathbf{Z}/p^{s-1}\mathbf{Z})$, so that $x_\delta \in$ Kernel α implies $\pi^*\theta^r(z) = 1$. But π^* is monic (compare the discussion following the statement of Proposition

1.5), so that $\theta'(z) = 1$ and hence $z \in \text{Kernel } \alpha$ by (1.4). Finally, the induction hypothesis implies $z = 0$, so that $x = x_\delta$ is also zero.

If $G = \mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/4\mathbf{Z}$, we can prove a stronger result than Theorem II.

PROPOSITION 1.10. *If KSph is the K-theory defined using stable spherical fibrations, and the $G = \mathbf{Z}/2\mathbf{Z}$ or $\mathbf{Z}/4\mathbf{Z}$, then the composite*

$$KO(BG) \rightarrowtail KTop(BG) \rightarrowtail KSph(BG)$$

is a monomorphism.

Proof of 1.10. For $G = \mathbf{Z}/2\mathbf{Z}$ this is a well-known consequence of the J-group calculations for real projective spaces [1, (6.3), p. 169]. For $\mathbf{Z}/4\mathbf{Z}$ it suffices to prove that the only element in $R(\mathbf{Z}/4\mathbf{Z})$ of the form $at + bt^2$ with $b \geq 0$ mapping trivially into each group $\tilde{J}(S^{2n-1}/(\mathbf{Z}/4\mathbf{Z}))$ (see [3]) is zero (write L for this lens space). For $2IO(\mathbf{Z}/4\mathbf{Z})$ lies in the subgroup generated by the images of $t - 1$ and $t^2 - 1$, and Proposition 1.5 shows $IO(\mathbf{Z}/4\mathbf{Z})^\wedge$ has no 2-torsion. In the first place, $a = 0$ follows by considering the induced bundle over each $\mathbf{R}P^{2n-1}$ and using the calculations for $\tilde{J}(\mathbf{R}P^{2n-1})$ as before. Assume bt $(b \geq 0)$ maps to zero in $\tilde{J}(L)$ for each n. Then by [1, (5.8), p. 158], there exists $\xi_n \in \tilde{K}(L^{2n-1})$ so that

$$\rho^5(bt^2) = \frac{5^b \psi^5(1 + \xi_n)}{1 + \xi_n}.$$

But $R(\mathbf{Z}/4\mathbf{Z}) \rightarrow K(\mathbf{Z}/4\mathbf{Z})$ is onto [3, p. 103], and thus ψ^5 is the identity. Hence by [1, (5.9), p. 159] we have $(3 + 2t^2)^b = 5^b$ in $K(L)$ for each L; it follows that the same holds in $R(\mathbf{Z}/4\mathbf{Z})$. To show this is impossible if $b > 0$, map $R(\mathbf{Z}/4\mathbf{Z})$ to the Gaussian integers sending t to i.

The remainder of the proof of sufficiency in Theorem II is a fairly straightforward modification of [3]. We first dispose of the case where G is *cyclic* of order not divisible by 8. Let $\{G_\lambda\}$ be the set of Sylow subgroups of G; then inclusions i_λ: $G_\lambda \subseteq G$ give rise to a commutative diagram of the following form:

$$\begin{tikzcd} KO(BG) \arrow[r, "\alpha_G"] \arrow[d, "\oplus Bi_\lambda^*"] & KTop(BG) \arrow[d] \\ \oplus KO(BG_\lambda) \arrow[r, "\oplus\alpha_\lambda"] & \oplus KTop(BG_\lambda) \end{tikzcd} \tag{1.11}$$

By the immediately preceding discussion, $\oplus \alpha_\lambda$ is monic; hence it suffices to check that $\oplus Bi_\lambda^*$ is monic. But it is immediate from the calculations of [3, p. 48] that the map

$$I(G)^k/I(G)^{k+1} \rightarrow \oplus I(G_\lambda)^k/I(G_\lambda)^{k+1}$$

is an isomorphism for every $k > 0$, and hence $\oplus Bi_\lambda^*$ is in fact an isomorphism. Finally, assume G is arbitrary and let $\{G_\lambda\}$ run through the cyclic subgroups of G. Another commutative diagram of type (1.11) arises for this choice of $\{G_\lambda\}$ and we already know that $\oplus\alpha_\lambda$ is monic. Therefore it suffices to show that $\oplus Bi_\lambda^*$ is monic; but this is an immediate consequence of [3, Thm. 7.2, p. 46, and Lemma 8.3, p. 48] and [4, §4].

Necessity. If α_G is not monic for $G = \mathbf{Z}/8\mathbf{Z}$, the general case will follow by taking induced representations. To see this, first notice there is a commutative diagram of the following form $(H = \mathbf{Z}/8\mathbf{Z})$

$$\begin{tikzcd} RO(BH)^\wedge \arrow[r, "\cong"] \arrow[d, "\text{ind.}"'] & KO(BH) \arrow[r] \arrow[d, "\text{transfer}"'] & KTop(BH) \arrow[d] \\ RO(BG)^\wedge \arrow[r, "\cong"] & KO(BG) \arrow[r] & KTop(BG) \end{tikzcd}$$

because (i) the induction map corresponds to the transfer of the finite covering $G/H \subseteq BH \rightarrow BG$ [21] (ii) By Boardman and Vogt[5], the map $KO \rightarrow KTop$ is a natural transformation of cohomology theories. Since H is central, the completed induction map is monic ($i^* i_* = [G:H]$ for representations and the completions are torsion free). Hence in this case an example for $H = \mathbf{Z}/8\mathbf{Z}$ can be induced up to G.

Let A: $BSO \to F/O$ be a solution to the Adams conjecture as in [14, §4] or [8], and consider the following commutative diagram of spaces localized at 2.

$$\begin{tikzcd} & F/O_{(2)} \arrow[r, "\tau"] \arrow[d] & F/Top_{(2)} \arrow[d] \\ BSO_{(2)} \arrow[ur, "A"] \arrow[r, "\psi^3-1"] & BO_{(2)} \arrow[r] & BTop_{(2)} \end{tikzcd}$$

Since $F/Top_{(2)}$ is homotopically a product $\Pi_{k>0}K(\mathbf{Z}_{(2)}, 4k) \times K(\mathbf{Z}/2\mathbf{Z}, 4k+2)$, the map τA corresponds to a sequence of cohomology classes $\alpha_j \in H^{2k}(BSO; \mathbf{Z}_{(2)}$ or $\mathbf{Z}/2\mathbf{Z})$; explicit formulas are given in [8, 14], but we shall not need them.

Let f_0: $BG \to BO$ be given by the standard representation ρ of G in SO_2, and let $f_1 = f_0\varphi$, where φ is induced by multiplication by 3 in G. Then φ induces the identity on $H^{2k}(BG; \mathbf{Z}_{(2)}$ or $\mathbf{Z}/2\mathbf{Z})$, and therefore $f_0 \simeq f_1$. Consequently the virtual representations $(\psi^3-1)\rho$, $\psi^3(\psi^3-1)\rho$ yield the same element in $KTop(BG)$. But $\psi^9 = 1$ on $KO(BG)$, and therefore it follows that $2(\psi^3-1)\rho$ is a nonzero element in the kernel of α_G. (I am indebted to G. Segal for the above argument which has replaced a more complicated one in a preliminary version).

The formulas of [8, 14] for α_j yield many other elements in the kernel of α_G for $G = \mathbf{Z}/2^r\mathbf{Z}$, $r \geq 3$. However, the application of this to the central problem of the present paper is unclear.

§2. PROOF OF THEOREM I AND GENERALIZATIONS

If G is a finite p-group (p odd), then Theorem I follows easily from Theorem II. For suppose V and W are topologically equivalent; then the balanced product bundles $EG \times_G V$ and $EG \times_G W$ determine the same element in $KTop(BG)$. Since $KO(BG) \to KTop(BG)$ is injective, these balanced products must determine the same element in $KO(BG)$; in other words, $V - W \in RO(G)$ lies in the kernel of the completion map $RO(G) \to RO(G)^{\wedge}$. But the latter is also injective for p-groups [3, Prop. 6.11, p. 45], and hence $V - W$ must be zero in $RO(G)$; i.e., V is linearly equivalent to W.

Assume now that G has p^s components where p is an odd prime (the case $s = 0$ is included, but in this case Theorem I follows from [13]). Since the equivalence class of a representation is determined by its character, two representations are equivalent if and only if they are equivalent on a dense union of subgroups. Thus we must verify that elements with order a power of p are dense in G if the latter has p^s components. This is a corollary of the following result, which is essentially due to G. Segal (compare [20, Remarks, p. 117]).

PROPOSITION 2.1. *Let G be a compact Lie group with c components, and let $g \in G$. Then there is a closed subgroup S containing g and satisfying the following conditions.*

(i) *The powers of some $s \in S$ are dense in S (i.e., S is topologically cyclic).*

(ii) *The number of components of S divides c^2.*

For the sake of completeness we shall give some additional groups for which topological equivalence equals linear equivalence. We begin with a general observation:

If topological equivalence equals linear equivalence for G, the same is true for $G \times \mathbf{Z}/2\mathbf{Z}$. (2.2)

Proof. Every representation of $G \times \mathbf{Z}/2\mathbf{Z}$ may be written as $V \cong (V_+ \otimes \mathbf{R}^+) \oplus (V_- \otimes \mathbf{R}^-)$, where $V_\mp$ is a G-representation and $\mathbf{R}^\pm$ is the one dimensional representation of $\mathbf{Z}/2\mathbf{Z}$ with $T(x) = \pm x$. If V and W are topologically equivalent, then the fixed point sets of $\mathbf{Z}/2\mathbf{Z}$ are topologically equivalent as G-representations; in other words $V_+ \oplus V_-$ and V_+ are topologically equivalent to $W_+ \oplus W_-$ and W_+ respectively. By the hypothesis on G the corresponding pairs must be linearly G-equivalent, and the linear equivalence of V and W over $G \times \mathbf{Z}/2\mathbf{Z}$ follows immediately from this and the above description.

COROLLARY 2.3. *Topological equivalence equals linear equivalence for each of the following classes of finite groups.*

(i) *All groups of the form $(\mathbf{Z}/2\mathbf{Z})^k$, $k \geq 1$*

(ii) *All cyclic groups $\mathbf{Z}/2p^s\mathbf{Z}$, where p is an odd prime and $s \geq 0$*

(iii) *All groups of order $2p^s$*

(iv) *All groups of the form $(\mathbf{Z}/4\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})^k$.*

Proof. (i) is a simple induction on (2.2), while (ii) follows from the Chinese Remainder Theorem and (2.2) (compare [12, 9δ]). (iii) is true because the representations agree on every cyclic subgroup by (ii) and Theorem I. (iv) will follow by induction from (2.2) if it is known for $\mathbf{Z}/4\mathbf{Z}$; but this is shown in [12].

This one final interesting class of examples that follows from the above discussion.

Topological equivalence implies linear equivalence for the orthogonal groups O_n, $n \geq 1$. (2.4)

Proof. Write $n = 2k + \epsilon$, with $\epsilon = 0$ or 1. Every element of O_n is conjugate to an element in $O_2{}^k \times O_1{}^\epsilon$, so by 2.3(ii) it suffices to check that elements of order $2p^s$ or p^s are dense in the latter. But consider the coordinates of a typical element g in $O_2{}^k \times O_1{}^\epsilon$; since every element of O_1 or O_2 not in the identity component has order 2, all coordinates of g not in an identity component have order 2. On the other hand, all coordinates in an identity component may be approximated by elements of order p^s (suitable p, s), and hence the desired density property holds.

Finally, the problem of classifying continuous representations up to topological equivalence makes sense even if G is an arbitrary locally compact group or the representation space is infinite-dimensional (or both). It would be interesting to have further information on the noncompact case, at least for some groups whose representations are well understood.

§3. TOPOLOGICAL INVARIANCE OF LINEAR SLICES

In this section we shall consider a question in [6, pp. 174–175]. Given a topological action on a manifold M^n with a linear slice at $x \in M$, is the equivalence class of the slice representation uniquely determined by x? In order to apply Theorem I and (2.2)–(2.4) to this problem we need the following.

Observation. Given two-G-modules V_1 and V_2, the following are equivalent.

(i) V_1 and V_2 are equivariantly homeomorphic;

(ii) There is a G-equivariant topological embedding of V_1 onto a neighborhood of the origin in V_2.

This is an immediate consequence of the Conical Orbit Structure Theorems in Bredon's book on compact transformation groups [6, Thms. II. 8.3, II. 8.4, pp. 98–104].

It follows that Theorem I and (2.2)–(2.4) go through with hypothesis (ii) replacing (i) (incidentally, Kister's identification of microbundles with bundles [11] yields the same conclusion). This observation implies a topological invariance theorem for slice representations [18, Thm. 1.6.5, Prop. 1.7.24] in differentiable G-manifolds.

THEOREM III. *Let M_1 and M_2 be smooth manifolds equipped with smooth actions of a compact Lie group G and satisfying the following conditions.*

(i) *M_1 and M_2 are equivariantly homeomorphic (let $h: M_1 \to M_2$ be an equivariant homeomorphism);*

(ii) *Every isotropy subgroup of G on M_1 (equivalently, M_2) satisfies the hypotheses of Theorem I, Corollary 2.3, or (2.4).*

Then the slice representations of $G_x = G_{h(x)} (= H)$ at x and $h(x)$ are equivalent.

Remark. In general, hypothesis (ii) is unnatural. On the other hand, it clearly holds if G is a finite group of order p^s or $2p^s$ (p an odd prime), or if $G = (\mathbf{Z}/2\mathbf{Z})^k$.

Proof. Let V_1 and V_2 be the linear representations of H on the tangent spaces of x and $h(x)$ respectively. Then there are invariant open neighborhoods U_1 and U_2 of x and $h(x)$ on which the action of G is equivalent to V_1 and V_2 respectively, with x and $h(x)$ corresponding to the origins; let g_i: $V_i \to U_i$ be the associated equivariant homeomorphisms. Then by continuity some set of the form $W_1 = g_1\{x \in V_1 |x| < \delta\}$ is mapped via h to a neighborhood of $h(x)$ in U_2; if we compose this with the G-homeomorphisms $W_1 \cong \{|x| < \delta\} \cong V_1$, $U_2 \cong V_2$, we get an embedding F: $V_1 \to V_2$ with $f(0) = 0$. Therefore, the generalization of Theorem I described above implies V_1 and V_2 are linearly equivalent.

The local representation of H on the tangent space of $y = x$ or $h(x)$ splits as $T_{G/H} \oplus S_y$, where $T_{G/H}$ is the representation of H on the tangent space of $[H] \in G/H$ and S_y is the slice representation at y (compare [18, *loc. cit.*]). Therefore $T_{G/H} \oplus S_x$ and $T_{G/H} \oplus S_{h(x)}$ are linearly equivalent by the preceding paragraph. The linear equivalence of S_x and $S_{h(x)}$ follows from this by inspecting the splitting of $V_1 \cong V_2$ into irreducible submodules.

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Purdue University